

Mapping between finite temperature classical and zero temperature quantum systems: quantum critical jamming and quantum dynamical heterogeneities

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Many electronic systems (e.g., the cuprate superconductors and heavy fermions) exhibit striking features in their dynamical response over a prominent range of experimental parameters. While there are some empirical suggestions of particular increasing length scales that accompany such transitions in some cases, this identification is not universal and in numerous instances no large correlation length is evident. Using a correspondence between dissipative classical and quantum many-body systems, we show that, even in the absence of imposed disorder, many continuum systems (and possible lattice counterparts) may exhibit zero-point “quantum dynamical heterogeneities” wherein the dynamics, at a given instant, is spatially non-uniform. Towards this end, we extend a known mapping between finite temperature classical Fokker-Planck systems and quantum systems at zero temperature to include general non-equilibrium dynamics. While the static length scales accompanying this phenomenon do not seem to exhibit a clear divergence in standard correlation functions, the length scale of the dynamical heterogeneities can increase dramatically. We furthermore show how a hard core bosonic system can undergo a zero temperature quantum critical metal-to-insulator-type transition with an extremely large effective dynamical exponent $z > 4$ that is consistent with length scales that increase far more slowly than the relaxation time as a putative critical transition is approached. We suggest ways to analyze experimental data in order to adduce such phenomena. Our approach may be applied to quenched quantum systems.

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I. INTRODUCTION

A prominent centerpiece in the understanding of numerous systems is Landau Fermi-liquid theory (LFL); this theory allows the understanding of phenomena such as conventional metals and low temperature He-3 liquids. LFL is centered on the premise that the low energy states of interacting electron systems may be captured by long-lived fermionic quasi-particles with renormalized parameters (e.g., effective masses that differ from those of the bare electron). The last three decades have seen the discovery of materials in which electronic behavior deviates from simple LFL. These “singular” or “non-Fermi liquids” (NFL) include the pseudo-gap region of the high-temperature cuprate superconductors and “heavy fermions” (in which, as befits their name, the effective electron mass becomes very large). While there are clear indications of changes in the dynamics in these systems—including putative quantum critical points^{1,2}—there is, in most cases, no clear experimentally measured length scale that exhibits a clear divergence. A quantum critical point is associated with a continuous phase transition at (absolute) zero temperature. Typically, this may occur in a system whose transition temperature is driven to zero by doping or the application of magnetic fields or pressure. Within a quantum critical regime, response functions follow universal power law scaling in both space and time. Specifically, at a quantum critical point, the effective infrared (IR) fixed point theory exhibits scaling

invariance in space-time: $t \rightarrow \lambda t$, $\vec{x} \rightarrow \lambda^{1/z} \vec{x}$ with a dynamical exponent z that can, depending on the theory at hand, assume various canonical values. Unlike classical critical points whose associated critical fluctuations are confined to a narrow region near the phase transition, quantum critical fluctuations appear over a wide range of temperatures above the quantum critical point. These fluctuations may generally lead to a radical departure of the system’s electronic properties from standard LFL type behavior. These features are anticipated to be common across many strongly correlated electronic systems and may be associated, in some electronic systems, with a change of Fermi surface topology³.

Many NFL systems exhibit numerous phases (including, quite notably, superconductivity). Indeed, competing orders and proliferation of multiple low energy states can lead to spatially non-uniform glassy characteristics⁴ and associated first order quantum transitions⁵. The length scales characterizing these electronic systems undergo a much milder change than the corresponding changes in the dynamics. All of this suggests an effective infrared (IR) fixed point quantum field theory is invariant under scaling in time but not in space—i.e., the effective dynamical exponent $z \rightarrow \infty$. Response functions such as those of the marginal Fermi liquid form describing cuprates near optimal doping show a marked frequency dependence but essentially no spatial momentum dependence. In this work, we will suggest how many quantum systems might exhibit very large effective dy-

namical exponents.

It is natural to first look elsewhere in physics where similar phenomena appear. One arena immediately comes to mind. In classical structural glasses there is a dramatic change in the dynamics as a supercooled liquid is quenched into a glass without the appearance of easily discernible large changes in measurable standard static length scales. While the ergodicity breaking that accompanies a glass transition cannot occur in a finite size system, it essentially mandates the appearance of a diverging static length scale⁶, but such divergent length scales generally do not simply manifest themselves in bare standard correlation functions. General correlation functions which may monitor subtle changes include the “point to set”⁷ correlations and others. Practically, in most instances⁸ no clear signatures of divergent length scales are easily seen in standard static two-point correlation functions.

A far more transparent growth in length scales is seen from four-point correlation functions that quantify the change in correlations as the system evolves in time. These correlation functions afford a glimpse into the length and time scaling which describe dynamical heterogeneities that characterize the spatially non-uniform rate of change or dynamics in the system. The length scale associated with these heterogeneities was seen to grow as the characteristic relaxation times increased.

We may use similar correlation functions to characterize strongly correlated electronic systems in which there are strongly discernible changes in the dynamics but no obvious experimentally accessible tools that point to accompanying divergent length scales in a general way⁹. To our knowledge, to date, dynamical heterogeneities (nor general static measures such as those of the point-to-set method) have not been systematically probed for in electronic systems nor has their existence been established as a matter of principle in quantum system. Initial ideas concerning non-uniform doping-driven heterogeneities were discussed in¹⁰. In this work, we flesh out the blueprint for a proof outlined, by one of us, in¹¹ and we will provide concrete “matter of principle” theoretical testimony to the emergence of quantum dynamical heterogeneities in *clean systems* and related properties in quantum many body systems. Some time after¹¹, the authors of¹² have further confirmed the existence of quantum dynamical heterogeneities in certain dissipative spin systems. In what follows, we focus on systems such as structural glasses or jammed systems with no external disorder.

As is well appreciated, such translationally invariant systems that are free of external disorder may, on their own, display non-uniform spatial patterns concomitant with interesting dynamical properties. For times far shorter than the equilibration times, the two-point auto-correlation function records dynamic fluctuations at all points \vec{x} in d dimensional space,

$$C(t) = \int d^d x \langle \delta\phi(\vec{x}, t) \delta\phi(\vec{x}, 0) \rangle, \quad (1)$$

forms an analogue of the Edwards-Anderson¹³ order parameter that appears in spin glasses. The spatial correlation amongst pair products of time separated products of fields (such as those in Eq. (1)) at different spatial sites is a four-point correlation function¹⁴ that attempts to measure cooperation

$$G_4(\vec{x} - \vec{y}, t) = \langle \delta\phi(\vec{x}, t) \delta\phi(\vec{x}, 0) \delta\phi(\vec{y}, t) \delta\phi(\vec{y}, 0) \rangle - C^2(t). \quad (2)$$

This four-point correlation function relates the dynamics at two different spatial points \vec{x} and \vec{y} ; due to translational invariance, this correlation function depends only on $(\vec{x} - \vec{y})$ and not on \vec{x} and \vec{y} separately.

II. REVIEW OF A RELATION BETWEEN VISCOUS CLASSICAL AND QUANTUM MANY BODY SYSTEMS AND A NEW GENERALIZATION

In order to illustrate how, as a matter of principle, the physics of such classical dissipative systems can appear in clean quantum systems at zero temperature, we employ and extend a mapping^{15,16,20–22} between classical dissipative systems and quantum many body systems to include general dynamical (including non-equilibrium) systems. Related work concerning dynamics in Rokhsar-Kivelson²³ systems also appears in²⁴. Below, we briefly review this mapping. We will then discuss some of its (new) consequences. Details of the derivation of our new result concerning a classical to quantum correspondence for the dynamics for general dissipative classical systems have been relegated to Appendix C.

The crux of the mapping between dissipative classical systems and many body bosonic theories^{15,16,20} is the realization that the equation of motion for a dissipative (or “Aristotelian”) classical system is a first order differential equation in time just as the Schrödinger equation is. Using this equivalence, systems obeying the Langevin equation,

$$\gamma_i \frac{d\vec{x}_i}{dt} = -\vec{\nabla}_i V_N(\vec{x}_1, \dots, \vec{x}_N) + \vec{\eta}_i(t), \quad (3)$$

[with i the particle index, γ_i the coefficients of friction, $\eta_i^\alpha(t)$ the Gaussian noise with

$$\langle \eta_i^\alpha(t) \eta_j^\beta(t') \rangle = 2T_{cl} \gamma_i \delta_{ij} \delta_{\alpha\beta} \delta(t - t'), \quad (4)$$

where T_{cl} is the effective temperature of the classical system, and, $\alpha, \beta = 1, 2, \dots, d$] can be exactly mapped^{15,16,20} onto a quantum many body system of bosons with effective mass $m_i = \gamma_i / (2T_{cl})$ at zero temperature which is governed by the Hamiltonian

$$\begin{aligned} H &= \sum_i \frac{1}{\gamma_i} \left[-T_{cl} \nabla_i^2 - \frac{1}{2} \nabla_i^2 V_N + \frac{1}{4T_{cl}} (\nabla_i V_N)^2 \right] \\ &\equiv \sum_i \frac{p_i^2}{2m_i} + \mathcal{V}_{\text{Quantum}}(\{\vec{x}\}). \end{aligned} \quad (5)$$

We will term the quantum system of Eq. (5) as the “dual quantum system” associated with the classical system of Eqs. (3, 4). The many body quantum potential $\mathcal{V}_{\text{Quantum}}(\{\vec{x}\})$ is constructed from the gradients of the classical potential energy V_N as in Eq. (5). Under this mapping,^{15–19} a dissipative classical system with a potential energy V_N that captures *repulsive hard core spheres* maps onto a quantum system at zero temperature with (as is apparent in the many body potential energy $\mathcal{V}_{\text{Quantum}}$) similar *dominant hard-core interactions* (augmented by soft sticky interactions)²⁰. Earlier work advanced^{15,17–20} the rudiments of this mapping and further suggested dynamical aspects that might follow from it. In the current work, we explicitly derive and prove a general and rather powerful relation between arbitrary correlation functions in dissipative classical systems with time varying potentials (necessary for our discussion of quenching) and relate these to corresponding correlation functions in zero-temperature quantum systems. In the below we provide a key result of this mapping. In Appendix C, we summarize, following^{15,17–20}, more detailed aspects of this mapping. We now proceed to set the stage for our new result and its consequences. [We should further note that although we will focus on the mapping from classical systems to corresponding quantum ones, it is also possible to go in the opposite direction and map quantum mechanical systems with known non-degenerate ground states onto classical dissipative systems (see appendix H).] In what follows, we consider a general classical two-point correlation function of the form

$$G_{\text{classical}}(t) = \langle \mathcal{O}(t) \mathcal{O}(0) \rangle, \quad (6)$$

where $\mathcal{O}(t)$ is *any* quantity at times $t \geq 0$. A notable extension of our work, presented in Appendix C, is that our mapping covers general non-equilibrium time dependent Hamiltonians in which only the initial (or the final) classical state is in thermal equilibrium at an initial (or final) temperature. In such a case, the corresponding dual quantum problem evolves unitarily with a time dependent Hamiltonian. Specifically, as we will elaborate on in Appendix C, we find a very simple and general result concerning a rather simple “complexification” of the time coordinate (or *Wick-type rotation*) in going from general classical correlation functions of the form of Eq. (6) to their corresponding bosonic counterparts. The final result is rather simple (Eq.(7)). For *any* quantity $\mathcal{O}(t)$ if the quantum correlation function $G_{\text{Quantum}}(\vec{q}, t)$ is evaluated with the quantum many body potential $\mathcal{V}_{\text{Quantum}}(\{\vec{x}\})$ of Eq. (A12) while the corresponding classical correlation function is computed for a system with a potential $V_{\text{classical}}(\{\vec{x}\})$ then a very simple relation exists,

$$R_{\text{Quantum}}(t) = i\Theta(t)(G_{\text{classical}}(it) - G_{\text{classical}}^*(it)). \quad (7)$$

Eq. (7) for the response function provides a new entry in the mapping between the finite temperature classical system of Eq. (3) governed by a potential V_N and its corresponding quantum zero-temperature dual with a

quantum many body potential energy $\mathcal{V}_{\text{Quantum}}(\{\vec{x}\})$ in the Hamiltonian of Eq. (5).

Eq. (7) is a key result of this work. In Appendix C, we provide a detailed derivation of Eq. (7). More general relation for correlation functions in systems with time dependent Hamiltonians (concerning evolving the system out of equilibrium classically or out of its ground state quantum mechanically) appear in Eqs. (C16, C17, C18). Typically, in glassy systems, the correlation function of Eq. (6) is a superposition of many decaying modes- This distribution of modes will manifest as a distribution of oscillatory modes in the corresponding dual quantum problem. In many cases, this will lead to zero temperature quantum dynamics of the dual system that, with additional oscillations, will emulate the finite classical dynamics. For instance, if, as it precisely occurs in viscous systems with overdamped dynamics (for which Eq. (3) applies), if for times $t > 0$,

$$G_{\text{classical}}(t) = A \exp[-(t/\tau)^a] \quad (8)$$

then, correspondingly for all of these positive times t (see Appendix E),

$$R_{\text{Quantum}}(t) = 2Ae^{-(\frac{t}{\tau})^a \cos \frac{\pi a}{2}} \sin \left[\left(\frac{t}{\tau} \right)^a \sin \frac{\pi a}{2} \right]. \quad (9)$$

With the aid of the general relation of Eq. (7), the quantum correlation function that corresponds to a general stretched exponential correlation function in the classical arena can be computed analytically and is, indeed, given by Eq.(9). In general, there are, of course, subtle issues relating to the analytic continuation of general functions to imaginary time. A trivial yet important particular realization of Eq. (7) is that static correlations ($t = 0$) are identical in the finite temperature classical and their corresponding quantum dual systems.

III. GLASSY QUANTUM DYNAMICS

Response functions in materials of changing plasticity, e.g., glass formers which as their temperature is lowered become progressively more viscous (as well as various electronic systems), suggest the presence of a distribution of local relaxation times that lead to, e.g., the canonical Cole-Cole or Cole-Davidson^{25,26} and similar forms as we briefly elaborate on. The response of a single attenuated mode to an initial impulse at time $t = 0$ scales as $g_{\text{single}} \sim \exp(-t/\tau)$ with τ the relaxation time; the Fourier transform of this response reads $g_{\text{single}}(\omega) = g_0/(1 - i\omega\tau)$. In systems with a distribution ($f(\tau')$) of relaxation events, the response functions are given by $\int d\tau' f(\tau') \exp(-t/\tau')$. Empirically, in dissipative plastic systems, relaxations scale as $(\exp[-(t/\tau)^a])$ with a power $0 < a < 1$ that leads to a “stretching” of the response function. This stretched exponential and other similar forms -such as the Cole-Cole (CC) and Davidson-Cole (DC) functions- capture quintessence of the distribution of relaxation times.^{25,26} With $g(\omega) = g_0 G(\omega)$,

where g_0 is a constant, the CC and CD forms correspond to different choices of G ,

$$\begin{aligned} G_{CC}(\omega) &= 1/[1 - (i\omega\tau)^a], \\ G_{DC}(\omega) &= 1/[1 - i\omega\tau]^b. \end{aligned} \quad (10)$$

The parameters a and b qualitatively emulate the real-time stretching exponent c . This distribution of relaxation times might be associated with different local dynamics (dynamical heterogeneities) to which we will turn to shortly in subsection III A. Fluids may fall out of equilibrium when cooled at a sufficiently rapid rate (so-called “super-cooling”) to become “glassy” at sufficiently low temperature. As liquids are supercooled, their characteristic relaxation times and viscosity may increase dramatically. There are several time scales that govern the dynamics of supercooled liquids. The so-called “ α relaxation” is associated with cooperative motion and leads to a pronounced rise of the viscosity (especially so in the “fragile” glass-formers). Empirically, in classical supercooled liquids at a temperature T_{cl} , the α relaxation times follow the Vogel-Fulcher-Tammann form²⁷,

$$\tau(T_{cl}) = \begin{cases} \tau_0 e^{\Delta/(T_{cl}-T_0)} & \text{for } T > T_0, \\ \infty & \text{for } T \leq T_0. \end{cases} \quad (11)$$

Here, T_0 is the temperature of the classical system at which the relaxation times (if Eq. (11) is precise) will truly diverge and Δ is an energy scale. Mode coupling theory²⁹ and numerous other theories might similarly capture aspects of the increase in the α relaxation time. By a trivial application of our result of Eq. (7), all of these finite temperature classical forms have quantum zero-temperature counterparts. Slower dynamics also appear as the system density increases. Several works, e.g.²⁸, suggest that relaxation times are a function of a composite quantity involving both density and temperature. It should be noted that all of our derivations start from Eq. (3). When examining the quantum dual to empirical forms describing classical liquids, the bare viscosity of the ambient liquid η appearing in these equations of motion may, in principle, be allowed to change as the temperature (and density) are varied. These may appear in addition to changes in T_{cl} and V_N (capturing, e.g., changes in the density). In classical simulated liquids, the bare viscosity may be kept constant.

A. Quantum dynamical heterogeneities

Our key result of Eq. (7) provides a natural bridge between classical and quantum correlations between disparate operators $\{\hat{\mathcal{O}}(t)\}$. The remainder of this work will largely focus on consequences of this relation. From it, correlation times $\{\tau_\alpha\}$ that appear in the classical problem will rear their head in the quantum problem (and vice versa).

We now focus on an intriguing aspect of classical glasses which by virtue of the relation of Eq. (7) [as alluded to in¹¹] leads to the appearance of new dynamical

correlations in quantum systems. Disorder free models for classical glass formers (including various simulated quenched systems such as those endowed with various classical potentials V_N (e.g.,³¹) that do not permit simple crystalline orders) are known to exhibit “dynamical heterogeneities” (DH)- a non-uniform distribution of local velocities³⁰ with the location of the more rapidly moving particles changing with time. By invoking Eq. (7), we see that *Quantum Dynamical Heterogeneities* (QDH)¹¹ appear in their corresponding zero temperature quantum counterparts. That is, in disorder free quantum systems derived (via Eq. (5)) from the corresponding classical systems, zero point dynamics is spatially non-uniform.

The presence of DH is seen by numerous probes³⁰. One often used metric is that of the four-point correlations of Eq. (2) in various guises. These correlation functions are of the form of Eq. (6) with $\mathcal{O}(t)$ denoting the overlap between fields ϕ when these are separated in time,

$$\mathcal{O}_{\vec{q}}(t) = \phi_{\vec{q}}(t)\phi_{-\vec{q}}(0) - \langle\phi_{\vec{q}}(t)\rangle\langle\phi_{-\vec{q}}(0)\rangle, \quad (12)$$

with \vec{q} any wave-vector. When Eq. (12) is substituted into Eq. (6), the Fourier space correlation functions (denoted $S_4^{\text{classical}}(\vec{q}, t)$ below) typically have an Ornstein-Zernicke type or similar related forms, e.g.,³²

$$S_4^{\text{classical}}(\vec{q}, t) = \frac{\chi_4(t)}{1 + q^2\xi_4(t)^2}, \quad (13)$$

with the length scale $\xi_4(t)$ representing the size of the typical dynamical heterogeneities when the system is examined at two times separated by an interval t . The four-point susceptibility is simply $\chi_4(t) = \int dx G_4(x, t)$. The key feature of Eq. (13) is that all of the q dependence has been relegated to a Lorentzian form while χ_4 and ξ_4 are otherwise general time dependent functions. We may next invoke Eq. (7) to generate the quantum counter-part of Eq. (13) (or of any other related form) and Fourier transform to real space to obtain, in the notation of Eq. (2), the spatial bosonic correlation function $G_4^{\text{Quantum}}(\vec{x}-\vec{y}, t)$ associated with the potential $\mathcal{V}_{\text{Quantum}}$ of Eq.(A12). The Fourier integral will be dominated by momentum space poles at $q = \pm i\xi_4^{-1}$. It is clear that in employing the transformation of Eq. (7), $G_4^{\text{Quantum}}(\vec{x}-\vec{y}, t)$ will exhibit exponential decay with the very same correlation length ξ_4 that is present in the classical system. This affords a direct proof of the dynamical length scales ξ_4 in *all zero temperature quantum counter-parts* (given by Eq. (5)) to any dissipative classical system that is known to exhibit these (and there are numerous known classical systems that exhibit dynamical heterogeneities³⁰).

Following our mapping, an exponentially decaying real space 4-point correlator ($r = |x-y|$) in the classical problem ($\exp(-r/\xi_4)$) will lead to an oscillatory decay in the quantum dual. Specifically, if the four-point correlation length in the classical problem diverges as $\xi_4 \sim \tau^{1/z}$ then in the corresponding quantum dual system the correla-

tion function of Eq. (2) will be given by

$$G_{4,\text{Quantum}}(r, t) \sim \sin\left(\frac{r}{\tau^{1/z}} \sin \frac{\pi}{2z}\right) \times \exp\left[-\frac{r}{\tau^{1/z}} \cos \frac{\pi}{2z}\right]. \quad (14)$$

B. Rapidly increasing time scale with concomitant slowly increasing length scales in quantum glasses

There is a proof that a growing static length scale must accompany the diverging relaxation times of glass transitions⁶. Some evidence has indeed been found for growing correlation lengths (static and those describing dynamic inhomogeneities)^{36–38}. As we noted earlier in this work, correlation lengths were studied via “point-to-set” correlations^{7,39} and pattern repetition size⁴⁰. Other current common methods of characterizing structures include (a) Voronoi polyhedra,^{41–43} (b) Honeycutt-Andersen indices⁴⁴, and (c) bond orientation⁴⁵: all centering on an atom or a given link. More recent approaches include graph theoretical tools⁴⁶. Notwithstanding current progress, it is fair to say that currently most “natural” textbook type length scales do not increase as dramatically as the relaxation time does when a liquid is supercooled and becomes a glass.

It is worthwhile to highlight that one of the most pertinent naturally increasing length scales is that associated with the typical size of the dynamical heterogeneity (i.e., ξ_4 of Eq. (13)). Similar to other measures, this typical length scale does not increase as rapidly as the characteristic relaxation time does as the glass transition is approached. Recent work for a three-dimensional bi-disperse repulsive glass⁴⁷ [with the pair potential $V(|\vec{x}|) = \epsilon(\sigma_{ab}/r)^{12}$ between two particles (a, b) of two possible types ($(a, b) \in 1, 2$) with $\sigma_{ab} = (\sigma_a + \sigma_b)/2$ and $\sigma_2/\sigma_1 = 1.2$], suggests that

$$\tau \sim \exp(k\xi_4^\theta), \quad (15)$$

with $\theta \simeq 1.3$ and k a constant. (An alternate assumed algebraic form $\tau \sim \xi_4^z$ leads to an extremely large dynamical exponent $z \simeq 10.8$.) In these cases, the dynamics changes dramatically with little notable change in the spatial correlation length.

C. Quantum Critical Jamming

The mapping between dissipative classical and quantum systems also suggests a new quantum critical point in related systems. The classical jamming transition^{48–59} of hard spheres/disks from a jammed system at high density to an unjammed one with spatially heterogeneous motion at lower densities is a continuous transition with known critical exponents, both static^{52,53} and dynamic⁵⁹. In most solids, the ratio of the shear to bulk modulus is a large number like 1/3. However at the jamming threshold, this ratio tends to zero. This renders the system very

soft to shear stresses. This is one of the things that makes jamming different from other solids⁴⁹. By our mapping, the classical jamming transition has a quantum analog. Replicating the mapping of the previous subsection (and, in particular, Eq. (7) therein), we may derive an analog quantum system harboring a zero temperature transition with similar critical exponents. The classical zero temperature critical point (“point J”)^{48,50} may rear its head anew in the form of *quantum critical jamming* (at a new critical point—“Quantum point J”) in bosonic systems. A schematic of the phase diagram of the associated quantum system is depicted in Fig. 1.

We may ascertain dynamical exponents from those reported for the classical jamming system⁵⁹. The classical low temperature system ($T_{cl} \rightarrow 0$) maps, according to Eq. (5), onto a zero temperature quantum system in its large mass limit. Bosons of infinite mass are not trivial due their statistics. Specifically, for a classical system of mono-disperse soft spheres with a repulsive force that is linear in the amount of compression, it was found that the correlation length ξ and relaxation time τ scale⁵⁹ as

$$\begin{aligned} \xi &\sim (\rho_J - \rho)^{-0.7}, \\ \tau &\sim (\rho_J - \rho)^{-3.3}. \end{aligned} \quad (16)$$

In Eq. (16), ρ denotes the density with ρ_J being the critical density at the jamming transition. Eq. (16) describes how the spatial and time scales diverge as the density is increased and approaches (from below) the density at the jamming transition. The correlation length in the jammed systems is set by the scale at which the number of surface zero modes is balanced by bulk effects. Taken together, these imply that, on approaching the transition, the relaxation time increases much more rapidly than the correlation length, $\tau \sim \xi_4^z$ with a very large effective dynamical exponent $z \simeq 4.6$. By use of Eq. (7), the same behavior is to be expected for the quantum system governed by the corresponding quantum potential $\mathcal{V}_{\text{Quantum}}$. The classical (and thus quantum) jamming exponents are the same in two and three-dimensions. It may be remarked that a similar dynamical exponent was found for a Bose glass model suggested to describe vortex line in high temperature superconductors⁶⁰. In physical terms, for charged bosons, the jamming transition constitutes a transition from a metallic system (when the system is unjammed and behaves as a fluid) to a jammed state (an insulator). We note earlier work rationalizing metal to insulator transitions in terms of electron pinning⁶³. In the bosonic jamming that we describe above, no pinning is present and the transition is driven by particle interactions.

As in earlier sections, we see that time scales increase far more precipitously than spatial correlation lengths.

One of the hallmarks of jammed systems is that the spectral density of vibrational excitations, $D(\omega)$ is constant at the jamming threshold^{53,61}. (In a Debye solid $D(\omega) \sim \omega^2$.) This is independent of potential, dimension of the system and size of the system. As one goes away

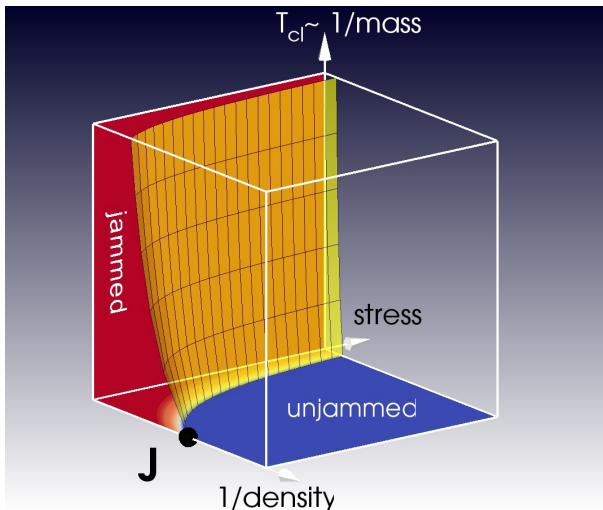


FIG. 1. (Color online.) The phase diagram of the zero temperature quantum jamming transition with line of J points. The phase boundaries and axis were formed by employing the phase diagram of the classical system⁴⁹ and examining the duality between the classical and quantum system- i.e., comparing the parameters in the quantum system of Eq. (5) with the classical system defined by Eqs. (3,4).

from the jamming threshold, $D(\omega)$ has a plateau down to a frequency ω^* . Below ω^* , the density drops and starts to look more like a Debye solid. This frequency ω^* veers to zero on approaching the transition. The number of low energy modes is set by the absence of constraints and Maxwell counting arguments. This constant density of states implies an enormous increase in the low frequency excitations. These can be measured by looking at the trajectory of a single particle and Fourier transforming its motion. One can also look at the spatial extent of these modes. They too are different from what one normally gets. They are quasi-localized (or resonant) at low frequency below ω^* . Above ω^* they are extended but still do not look at all like plane waves.⁴⁹ The results concerning the density of the modes hold unchanged for the dual quantum system. This is so as the dynamical matrix D is formed by the second derivatives of the potential V_N relative to the displacements. Thus, the same

statement about mode density of states that appear in classical jammed system is reflected verbatim in the classical potential V_{Nf} which we use to construct the quantum many body potential $V_{\text{Quantum}}(\{\vec{x}\})$ from Eq. (5). Any appearance of zero energy (bulk or surface) modes in the classical system will identically hold also in the quantum system.

IV. LATTICE SYSTEMS

Thus far, we focused on continuum viscous classical systems which, as we have seen, mapped onto continuum bosonic systems. We briefly remark here on classical lattice systems which similarly exhibit dynamical heterogeneities and a jamming type transition. [In appendix H, we further briefly expand on our derivations for lattice systems.] Refs.^{64,65} studied, respectively, the 2DN3 and 3DN2 models on the square and cubic lattice models in $d = 2$ and $d = 3$ dimensions. In the square lattice N3 model, particles are endowed with hard core repulsive interactions that extend up to a distance of three steps on the lattice. Similarly, in the cubic lattice N2 model particles cannot be nearest neighbors nor next nearest neighbors (i.e., the repulsive hard-core interactions extend up to a distance of two steps on the lattice). Thus, similar to the continuum systems that we discussed earlier, these models may be regarded as that of classical hard core spheres. Following the mapping reviewed in section II, the quantum counterpart of such systems is that of dominant hard sphere interactions augmented by contact sticky interactions. In the classical systems, simulation starts^{64,65} with an infinitely fast quenching wherein particles are added whenever possible and diffuse otherwise; this process is halted when the desired density is reached. A clear increase was noted in the length scales that characterize the dynamical heterogeneity^{64,65}. The continuum jamming transition discussed earlier may have a lattice counterpart for Cooper pairs as we now elaborate on. A natural quantum counterpart to the N3 (N2) model is given by an extended Bose Hubbard^{66,67} type model with infinite hard core repulsions,

$$H = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + h.c.) + U \sum_i n_i (n_i - 1) + \sum_{ij} V_{ij} n_i n_j, \quad (17)$$

where $V_{ij} \rightarrow \infty$ for lattice sites i and j which are fewer than four (or three) steps apart and the onsite Hubbard repulsion U is divergent ($U \rightarrow \infty$) as well. The Hubbard term leads to a penalty only when there is a double or higher occupancy. Based on our considerations thus far, we expect to obtain the quantum bosonic counterpart to the classical jamming transitions found in the classical 2DN3 and 3DN2 models. This bosonic system may have all of the characteristics of the classical jammed system including dynamical heterogeneities and a large dynamical exponent z . For completeness, we briefly comment on

the difference between the lattice system of Eq. (17) and the “Bose glass” first introduced in⁶⁶. The Bose glass appears in the bare (i.e., that with $V_{ij} = 0$) disordered rendition of Eq. (17) with the general Bose Hubbard Hamiltonian (with general finite repulsion U) being further augmented by a local chemical potential term $-\sum_i \mu_i n_i$ wherein μ_i is a spatially non-uniform random quantity. By contrast, the lattice Hamiltonian of Eq. (17) as well as the continuum models that we discussed in earlier sections are free of disorder. The amorphous characteristics that these clean systems may exhibit are borne out of “self-generated” randomness⁶⁸—not randomness that is present in the parameters defining the system.

V. ELECTRONIC SYSTEMS WITH PAIRING INTERACTIONS

Up to now, building on and extending the mapping between classical dissipative systems and zero temperature bosonic theories, we focused on hard core bosons. We now turn to the ground states of Fermi systems. In particular, in this section, we will consider standard electronic systems with pairing interactions,

$$H = \sum_{\vec{k}, \sigma} \epsilon_{\vec{k}} c_{\vec{k}\sigma}^\dagger c_{\vec{k}\sigma} + \sum_{\vec{k}, \vec{l}} V_{\vec{k}, \vec{l}} c_{\vec{k}\uparrow}^\dagger c_{-\vec{k}\downarrow}^\dagger c_{-\vec{l}\downarrow} c_{\vec{l}\uparrow}, \quad (18)$$

where $\sigma = \uparrow, \downarrow$ is the spin polarization index and V_{kl} is the interaction strength between the Cooper pairs $|\vec{k} \uparrow; -\vec{k} \downarrow\rangle$ and $|\vec{l} \downarrow; -\vec{l} \uparrow\rangle$. As is well known (and is readily verified), the following Fermi bilinears

$$\begin{aligned} \bar{b}_{\vec{k}}^\dagger &= c_{\vec{k}\uparrow}^\dagger c_{-\vec{k}\downarrow}^\dagger, \\ \bar{b}_{\vec{k}} &= c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow}, \end{aligned} \quad (19)$$

corresponding to the creation/annihilation of Cooper pairs satisfy hard core Bose algebra. We next consider what occurs if, within the ground state, the occupancies of the single particle states are correlated inasmuch as the electronic states on which the standard pairing Hamiltonian of Eq. (19) operates can be created by applications of Cooper pair creation operators on the vacuum (i.e., if the ground state is invariant under the combined operations of parity ($\vec{k} \rightarrow -\vec{k}$) and time reversal ($\sigma \rightarrow -\sigma$)). When the ground state is strictly invariant under the combined effect of these symmetries, we may express the Hamiltonian of Eq. (18) as a bilinear in the hard core Bose operators,

$$H = \sum_{\vec{k}, \vec{l}} (2\epsilon_{\vec{k}} \delta_{\vec{k}, \vec{l}} + V_{\vec{k}, \vec{l}}) \bar{b}_{\vec{k}}^\dagger \bar{b}_{\vec{l}}. \quad (20)$$

The hard core (Fourier space) Bose algebra of the creation and annihilation operators [as, in particular, manifest in the relation $(\bar{b}_{\vec{k}}^\dagger)^2 = 0$ mandating that no more than one boson can occupy any given (Fourier space)

site] is identical to that of raising and lowering operators in the spin $S = 1/2$ system. Thus, a simple extension of the standard real-space Matsubara-Matsuda transformation⁶⁹ is given by

$$\begin{aligned} \bar{b}_{\vec{k}}^\dagger &\rightarrow S_{\vec{k}}^+, \\ \bar{b}_{\vec{k}} &\rightarrow S_{\vec{k}}^-. \end{aligned} \quad (21)$$

Substituting Eq. (21) into Eq. (20), we arrive at an XY model. In situations in which the band dispersion $\epsilon_{\vec{k}}$ is nearly flat (and may be omitted for fixed particle number), in determining the ground state(s), we must only find the pairing V that affects pair hopping. Similar considerations apply in real-space when Cooper pairs are short ranged and may be replaced by real-space hard-core bosons. Hard core real space contact interactions correspond to uniform $V_{\vec{k}, \vec{l}}$ (independent of \vec{k} and \vec{l}) as in the BCS form for the pairing interactions. In such cases, whenever the system is dominated by hard core contact interactions between the bosonic Cooper pairs we see, replicating our analysis thus far, at zero temperature, that the system may undergo a jamming type transition between an itinerant and jammed phase at sufficiently high densities or pressure. In this case, it displays rapidly increasing relaxation times concomitant with spatial correlations that do not increase as dramatically as the relaxation times do on approaching this transition.

VI. GENERAL BOSONIC QUANTUM CRITICAL POINTS WITH CLASSICAL ANALOGS

Up to this point our focus has been on supercooled systems. Eq. (7) implies that also standard (non glassy or spin-glassy) classical transitions have corresponding zero temperature quantum analogs. Bona fide critical or discontinuous classical transitions have quantum duals according to the methods that we invoke. Ising type transitions have critical exponents z which are given by $z_{2D} \simeq 2$ and a somewhat smaller value in $D = 3$ dimensions. At the critical point of the liquid to gas transition, the system is in the Ising universality class. Thus, the transition from a quantum Bose gas to a fluid is governed by the same dynamical exponents.

VII. POSSIBLE IMPLICATIONS FOR EXPERIMENTAL DATA

We now, very briefly, turn to a discussion of possible data analysis of experiments. One of the main messages of our work is that classical physics associated with overdamped classical systems can rear its head in the quantum arena. Correspondingly, data analysis which has led to much insight in the study of classical glasses and other damped systems may be performed anew for quantum systems. A principal correlation function which we

focused on in this work has been that of the four-point correlation function of Eq. (2). This correlation function need not be directly measured in real-time. For instance, scanning tunneling spectroscopy (STS) data taken at different positions and bias voltages may provide a valuable conduit towards the evaluation of the four-point correlator when it is expressed as an integral over frequencies (or associated bias voltages). Rather trivially with $\phi(\vec{x}, V)$ denoting the local density of states at location x for a bias voltage V , and e^* the electronic charge, the corresponding four-point correlation function is given by

$$G_4(\vec{x} - \vec{y}, t) = \int dV_1 dV_2 dV_3 dV_4 \langle \delta\phi(\vec{x}, V_1) \delta\phi(\vec{x}, V_2) \delta\phi(\vec{y}, V_3) \delta\phi(\vec{y}, V_4) \rangle e^{ie^*t(V_1+V_3)} - C^2(t), \quad (22)$$

with the two-point auto-correlation function

$$C(t) = \int d^d x \int dV \langle \delta\phi(\vec{x}, V) \delta\phi(x, 0) \rangle e^{ie^*tV}. \quad (23)$$

As we have discussed in Section III A, quantum dynamical heterogeneities may be manifest in this correlation function via the standard relation of Eq. (13).

VIII. CONCLUSIONS

A central result of this work is the exact temporal correspondence of Eq. (7) that spans both equilibrium and non-equilibrium dynamics in general time dependent systems (so long as either the initial or the final state of the classical system is that of thermal equilibrium). This equality relates (i) the auto-correlation function of Eq. (6), for *any quantity* \mathcal{O} when evaluated for the classical dissipative system of Eq. (3) with a many body potential energy V_N , to (ii) the auto-correlation function of the very same corresponding quantum operator $\hat{\mathcal{O}}$ in a dual bosonic system governed by the Hamiltonian of Eq. (5). When fused with known results for dissipative classical systems, this extremely general equality immediately leads to numerous non-trivial effects which we introduced and readily proved as a matter of principle. These include:

- *Quantum dynamical heterogeneities* (QDH). We illustrated that similar to classical systems even in the absence of disorder, bosonic systems can, at zero temperature, exhibit spatially non-uniform zero-point motion. Of course, in translationally invariant systems, the average (time averaged) dynamics is uniform. However, at any given time, there are particles that move more rapidly than others. We suggested how experimental data may be analyzed to search for quantum dynamical heterogeneities in electronic systems.

- The length scale characterizing the zero temperature QDH, the four-point correlation length ξ_4 (a trivial analog of its classical counterpart) may increase as the dynamics of the clean Bose system becomes progressively

sluggish. However, albeit its rise, this length scale may increase much more slowly than the relaxation time. The far more rapid increase of the relaxation time as compared to readily measured length scales is a hallmark of many electronic systems. Cast in terms of quantum critical scaling (if and when it might be realized), the effective dynamical exponent z capturing the relation between correlation lengths and times is very large ($z \gg 1$). Other relations such as those of Eq. (15) may hold once they are established for viscous classical systems.

- Similar to classical systems, quantum systems may jam at high densities or pressure notwithstanding zero point motion. The character of the jamming transition in zero temperature quantum systems is identical to that of that of their corresponding classical finite temperature counterparts. As the classical systems exhibit a critical point at the jamming transition (at “point J”) so do their bosonic counterparts. As a result, we established the existence of a new quantum critical point—associated with a *quantum critical jamming* of a hard core Bose system. As in the other systems that we discussed, the characteristic relaxation time diverges more precipitously than the correlation length on approaching the transition (“Quantum point J”) with a large effective dynamical exponent $z \simeq 4.6$.

- The continuum theories that we predominantly focused on may have a broad applicability as continuum theories describe the same physics as their lattice renditions do in the vicinity of critical points. In Section IV, we discussed specific possible lattice renditions.

- The results that we derived for zero temperature bosonic theories suggest similar features in electronic systems. In some cases, as discussed in Section V, finding the ground states of interacting electronic systems can be cast in terms of a corresponding zero temperature hard-core Bose problem.

- The general mapping of Eq. (21) between hard core bosons to $S = 1/2$ spin systems (in either momentum (\vec{k}) or real (\vec{r}) space) along with the complementary relation for the z component of the spin,

$$[\bar{b}_{\vec{k}}^\dagger \bar{b}_{\vec{k}} - 1/2] \rightarrow S_{\vec{k}}^z, \quad (24)$$

in the same space, allows us to derive similar results for certain spin $S = 1/2$ systems.

Thus, with the aid of the viscous classical-many body quantum correspondence of Eq. (7), we trivially established all of these results without the need to perform various standard and far more laborious computations for quantum systems.

Other possible extensions of our results include quantum turbulence from known results on the dynamics in classical fluids that exhibit turbulence, the relation between localization (or caging) in classical systems and their corresponding quantum counterparts. We may similarly examine disordered systems- for a random classical potential V_N , the corresponding quantum potential $\mathcal{V}_{\text{Quantum}}$ is also random.

As we will further detail in the appendix (see Eq. (A20)), if we are given a known quantum ground state, we may find the corresponding effective classical potential. With the aid of calculations on how the correlation functions of the classical system depend on time as parameters in the classical potential are varied, we may then determine the corresponding time dependent correlation functions of the many quantum system. That is, we need not always find corresponding quantum systems to classical systems; the Fokker-Planck mapping also enables to go in the opposite direction from quantum systems to classical ones.

Numerous related extensions may be considered. For instance, we may consider magnetic and other systems in which fermionic degrees can be formally integrated out leaving only effective bosonic degrees of freedom. Consequences for the Ward identity relating four-point with two-point correlation functions may be considered. The Langevin equation may be re-examined for single vortex crossing of a narrow superconducting wire at finite temperature to derive the mapping for the quantum dual at absolute zero temperature.^{73,74} This would offer an alternative path for exploring the viability of quantum phase slips in nanowires.

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Appendix A: The Fokker Plank of the dissipative classical system and its relation to a quantum Hamiltonian

In what follows, we highlight features of the mapping^{15,16,20} between classical dissipative systems and bosonic systems. We first set the preliminaries following¹⁶. Given the initial vector x_0 of the coordinates of all particles at time $t = t_0$, the time dependent probability distribution $\mathcal{P}(\vec{x}, t; \vec{x}_0, t_0)$ for the correspond-

ing position vectors $\{\vec{x}(t)\}$ at time t is given by

$$\mathcal{P}(\{\vec{x}\}, t; \{\vec{x}_0\}, t_0) = \langle \prod_{i=1}^N \delta[\{\vec{x}_i(t) - \vec{x}_i\}] \rangle_{\{\vec{\eta}\}, \{\vec{x}_0\}} \quad (\text{A1})$$

where $\langle - \rangle_{\{\vec{\eta}\}, \{\vec{x}_0\}}$ denotes the average over the random noise η (which we will take to be the Gaussian white noise of Eq. (4)) given that initially, at time $t = t_0$, the particle coordinates were $\{\vec{x}_0\}$. The average of a general function $\mathcal{O}(\{\vec{x}(t)\})$ is then

$$\begin{aligned} \int d^{dN} x \mathcal{P}(\{\vec{x}\}, t; \{\vec{x}_0\}, t_0) \mathcal{O}(\{\vec{x}\}) \\ = \langle \mathcal{O}(\{\vec{x}(t)\}) \rangle_{\{\vec{\eta}\}, \{\vec{x}_0\}}. \end{aligned} \quad (\text{A2})$$

It is convenient to write $\mathcal{P}(\{\vec{x}\}, t; \{\vec{x}_0\}, t_0)$ in a Dirac notation as

$$\mathcal{P}(\{\vec{x}\}, t; \{\vec{x}_0\}, t_0) = \langle \{\vec{x}\} | \hat{P}(t, t_0) | \{\vec{x}_0\} \rangle. \quad (\text{A3})$$

Time translation invariance and the Markov property of these probabilities,

$$\begin{aligned} \int d^{dN} x' \langle \{\vec{x}\} | \hat{P}(t, t') | \{\vec{x}'\} \rangle \langle \{\vec{x}'\} | \hat{P}(t', t_0) | \{\vec{x}_0\} \rangle \\ = \int d^{dN} x' \mathcal{P}(\{\vec{x}\}, t; \{\vec{x}'\}, t') \mathcal{P}(\{\vec{x}'\}, t'; \{\vec{x}_0\}, t_0) \\ = \mathcal{P}(\{\vec{x}\}, t; \{\vec{x}_0\}, t_0), \end{aligned} \quad (\text{A4})$$

imply that

$$\hat{P}(t, t_0) = \mathcal{T} e^{-\int_{t_0}^t H_{FP}(t') dt'}, \quad (\text{A5})$$

with a ‘‘Fokker-Planck’’ Hamiltonian H_{FP} and where \mathcal{T} is the time ordering operator. We now return to our particular classical to quantum mapping. The summary below closely follows this mapping as presented by Biroli et al.²⁰. In what follows, we set

$$P = \hat{P}(t, t_0) | \{\vec{x}_0\} \rangle. \quad (\text{A6})$$

For the classical dissipative system of Eq. (3), the probability distribution $P(\{\vec{x}\})$ evolves according to the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -H_{FP} P, \quad (\text{A7})$$

where the Fokker Planck operator is

$$H_{FP} = - \sum_i \frac{1}{\gamma_i} \vec{\nabla}_i \cdot \left[\vec{\nabla}_i V_N + T_{cl} \vec{\nabla}_i \right], \quad (\text{A8})$$

with T_{cl} being the temperature of the classical system [setting the noise strength in Eq. (4)]. Eq. (A8) follows from a direct differentiation of Eq. (A1) while invoking Eq. (3) for the derivatives of the coordinates $\{\vec{x}_i(t)\}$ in the argument of the delta functions and performing short time averages. (Thus, this equation and our results pertain to systems in which the dynamics is sufficiently slow such that short time averages over the noise

η at fixed temperature are sensible.) A detailed derivation of the Fokker-Planck equation for this and more general Langevin processes appears in many excellent textbooks, e.g.,⁷⁰. The operator H_{FP} is non-Hermitian. Each eigenvalue is generally associated with differing left and right eigenvectors. The Fokker-Planck equation can be mapped into a Hermitian Hamiltonian by⁶²

$$H = e^{V_N/(2T_{cl})} H_{FP} e^{-V_N/(2T_{cl})}, \quad (\text{A9})$$

if the second derivatives are exchangeable, $\vec{\nabla}_i \vec{\nabla}_j V_N = \vec{\nabla}_j \vec{\nabla}_i V_N$.⁷¹ A direct substitution leads to the quantum many body Hamiltonian of Eq. (5). Note that, thus far, we have allowed V_N to be completely general. This potential energy may include one body interactions (i.e., coupling to an external source), pair interactions between particles, and three- and higher-order particle interactions. A key point that we will further invoke later is that the transformation of Eq. (A9) leading to a Hermitian quantum Hamiltonian can be trivially performed at any given time slice when V_N and T_{cl} are, generally, time dependent. It is worth highlighting that in non-equilibrium time dependent classical systems, the temperature T_{cl} is set by the time dependent noise amplitude (following Eq. (4)). We consider what specifically occurs when the classical potential energy in Eq. (3) is the sum of pairwise interactions,

$$V_N(\{\vec{x}\}) = \frac{1}{2} \sum_{i \neq j} V_{ij}(\vec{x}_i - \vec{x}_j). \quad (\text{A10})$$

For such systems, the quantum many body Hamiltonian of Eq. (5) explicitly contains an effective potential which is the sum of two and three body interactions,

$$\begin{aligned} \mathcal{V}_{\text{Quantum}}(\{\vec{x}\}) &= \sum_i \frac{1}{\gamma_i} \left[-\frac{1}{2} \nabla_i^2 V_N + \frac{1}{4T_{cl}} (\vec{\nabla}_i V_N)^2 \right] \\ &= -\frac{1}{2} \sum_{i \neq j} \frac{1}{\gamma_i} \nabla_i^2 V_{ij} + \sum_{i,j \neq i; j' \neq i} \frac{\vec{\nabla}_i V_{ij} \cdot \vec{\nabla}_i V_{ij'}}{4T_{cl} \gamma_i}. \end{aligned} \quad (\text{A11})$$

For a given classical two body potential in d dimensions which is both translationally and rotationally invariant, $V(\vec{x}) = V(|\vec{x}|)$, the resulting quantum potential energy is given by (as in²⁰ yet now trivially extended to include general T_{cl}),

$$\begin{aligned} \mathcal{V}_{\text{Quantum}}(\{\vec{x}\}) &= \frac{1}{2} \sum_{i \neq j} v_{\text{Quantum}}^{\text{pair}}(\vec{x}_i - \vec{x}_j) \\ &\quad + \sum_{i,j \neq i; j' \neq i} v_{\text{Quantum}}^{\text{3-body}}(\vec{x}_i - \vec{x}_j, \vec{x}_i - \vec{x}_{j'}); \\ v^{\text{pair}}(\vec{x}) &= -\nabla^2 V(\vec{x}) + \frac{1}{2T_{cl}} [\vec{\nabla} V(\vec{x})]^2 \\ &= -\frac{d-1}{r} V'(r) - V''(r) + \frac{1}{2T_{cl}} [V'(r)]^2; \\ v^{\text{3-body}}(x, x') &= \frac{1}{4T_{cl}} \vec{\nabla} V(x) \cdot \vec{\nabla} V(x') \\ &= \frac{1}{4T_{cl}} \frac{\vec{x}}{r} \cdot \frac{\vec{x}'}{r'} V'(r) V'(r'), \end{aligned} \quad (\text{A12})$$

with $r = |\vec{x}|$. For a classical potential $V(r) = V_0 \exp(-\lambda[(r/\sigma)^2 - 1])$ the corresponding pair term in Eq. (A12) is

$$v^{\text{pair}}(r) = \frac{2\lambda d - 4\lambda^2 r^2}{\sigma^2} V(r) + \frac{2\lambda^2 r^2}{T_{cl} \sigma^4} [V(r)]^2. \quad (\text{A13})$$

In the limit $\lambda \rightarrow \infty$, the classical system corresponds to that of hard spheres where σ is the diameter of the hard sphere and the quantum potential of Eq. (A13) similarly exhibits a dominant hard sphere repulsion (augmented by an attractive potential at the sphere boundaries that is of range $1/\lambda$).

For general $V_N(\{\vec{x}\})$, the Fokker-Planck operator of Eq. (A8) has non-negative eigenvalues^{15,16}. For any time independent H_{FP} , the zero eigenvalue state—i.e., the ground state—which according to Eq. (A7) corresponds to a stationary (time independent) probability distribution P . This is the equilibrium Boltzmann distribution

$$P^{\text{equil}}(\{\vec{x}\}, t) = \frac{1}{Z_N} e^{-\beta V_N(\{\vec{x}\})}, \quad (\text{A14})$$

with Z_N the partition function associated with $V_N(\{\vec{x}\})$ and $\beta = 1/T_{cl}$. This is readily rationalized by the following argument. For a finite size system, the linear eigenvalue equation

$$(H_{FP})_{bc} P_c = -\varepsilon_c P_c, \quad (\text{A15})$$

with the matrix row/column indices b and c denoting classical configurations, has a (finite size) matrix H_{FP} with positive off-diagonal elements and negative diagonal entries. Specifically, in Eqs. (A7,A15), the transition matrix H_{FP} has entries that relate the probabilities of going from state b to state c in a given (infinitesimal) time interval. If these states are different ($b \neq c$) then clearly $(H_{FP})_{bc} > 0$. The diagonal elements $(H_{FP})_{bb}$ provide the probabilities of “leaking out” of state b and going to all other states $c \neq b$. From all of this it follows that

$$(H_{FP})_{bb} = -\sum_{b' \neq b} (H_{FP})_{bb'} < 0. \quad (\text{A16})$$

Detailed balance, i.e., the fact that the probability of going from b to c is the same as that of going from c to b , asserts that

$$(H_{FP})_{bc} e^{-\beta E_b} = (H_{FP})_{cb} e^{-\beta E_c}. \quad (\text{A17})$$

Eqs. (A15, A17) illustrate that the Hamiltonian of Eq. (A9) is Hermitian. In this classical system of Eq. (3), the energies of the classical states E_c are simply given by $V_N(\{\vec{x}\})$ evaluated for the classical configurations c . With the aid of Eqs. (A16, A17), it is easy to see that the column vector $P_c^{\text{equil}} = Z_N^{-1} \exp(-\beta E_c)$ (i.e., the distribution of Eq. (A14)) is a null eigenvector of Eq. (A15). This probability eigenvector corresponds, of course, to the equilibrium Boltzmann distribution. The factor of Z_N^{-1} is inserted to ensure normalization of the classical

probabilities (for any eigenstate): $\sum_c P_c = 1$. Now, we can add a constant to the finite dimensional matrix

$$H_{FP} \rightarrow H_{FP} - \text{const.} \equiv H'_{FP}, \quad (\text{A18})$$

to generate a matrix $(-H'_{FP})$ that has all of its elements positive $(-H'_{FP})_{bc} > 0$. Specifically, to this end, in Eq. (A18) we can choose const. to be any constant larger than the sign inverted smallest off-diagonal element of $(-H_{FP})$, i.e., $\text{const.} > -\min_{b \neq c} \{H_{FP}\}_{bc}$. For such a positive matrix, we can apply the Perron-Frobenius theorem which states that the largest eigenvector of $(-H'_{FP})$ is non-degenerate and that eigenvector is the only eigenvector that has all of its elements positive with all other orthogonal eigenvectors having at least one negative element. Clearly, all of the eigenvectors of H_{FP} and H'_{FP} are identical with the corresponding eigenvectors of both operators merely shifted uniformly by a constant. With all of the above in tow, we see that P^{equil} corresponds to the largest eigenvector of $(-H'_{FP})$ and is thus also the largest eigenvector of $(-H_{FP})$. For a time independent H_{FP} , as P^{equil} was the null eigenvector of H_{FP} , it follows that all other eigenvalues of Eq. (A15), $\varepsilon > 0$, are positive and, according to Eq. (A7) and explicit earlier discussions evolve with time as $\exp(-\varepsilon t) \rightarrow_{t \rightarrow \infty} 0$. Thus, physically (as it to be expected) at long times the system attains its equilibrium configuration of P^{equil} . In the corresponding zero-temperature quantum problem, the dominant classical equilibrium state with a lowest energy. We will thus label it in Appendix C by $|G\rangle$. The transformation of Eq. (A9) relates the operators in the classical Fokker-Planck and zero temperature quantum problem to one another. The transformation for the right eigenvectors P of H_{FP} , which we explicitly denote below as $|- \rangle_{FP}$, to the eigenvectors of the quantum Hamiltonian H is trivially

$$|- \rangle_{FP} \rightarrow \exp(-V_N/(2T_{cl}))|- \rangle_{FP} = |- \rangle_{\text{Quantum}} \quad (\text{A19})$$

Similarly, the left eigenvectors $(\langle - |_{FP})$ of H_{FP} are to be multiplied by $\exp(V_N/(2T_{cl}))$ in order to pass to the left eigenstates of the quantum problem. (In the quantum problem defined by H , the left and right eigenstates are trivially related to each other by Hermitian conjugation.) Applying Eq. (A19) to the null right eigenstate of the Fokker Planck Hamiltonian of Eq. (A14), we see that the quantum eigenstate of H corresponding to this classical equilibrium state is given by

$$\Psi_0(\{\vec{x}\}) = \frac{1}{\sqrt{Z_N}} \exp(-\frac{1}{2T_{cl}} V_N(\{\vec{x}\})). \quad (\text{A20})$$

The prefactor in Eq. (A20) is set by the normalization of this quantum state. When comparing Eqs. (A14, A20) to one another, we see that this wavefunction is related to the classical equilibrium probability eigenvector by the appealing relation $\Psi_0(\{\vec{x}\}) = \sqrt{P^{\text{equil}}(\{\vec{x}\})}$. When $V_N(\{\vec{x}\})$ is symmetric under the interchange of particle coordinates, the resulting wavefunction may describe bosons. For two body interactions V_{ij} (Eq. A10) that

are symmetric under the permutations of i with j , the wavefunction of Eq. (A20) is symmetric under any permutations of the particles. Thus, the ground state wavefunction of Eq. (A20) is a Jastrow type wavefunction describing a bosonic system. Of course, generally, $V_N(\{\vec{x}\})$ can include not only two body terms but also single body contributions (local chemical potentials or fields) as well as three- and higher-body interactions. Although obvious, it is worth noting that if $V_N(\{\vec{x}\})$ (and thus the quantum Hamiltonian H) is invariant under any pairwise permutation P_{ij} , i.e., if $[H, P_{ij}] = 0$ then the symmetry of the initial wavefunction Ψ_0 (corresponding to the classical Boltzmann distribution for a system initially at an equilibrium at temperature T_{cl}) does not change as the system evolves with time (including general arbitrary H corresponding to classical variations in temperature and other parameters).

We complete this appendix with a brief discussion concerning a generalization of Eq. (A9). It is, of course, possible to write down a general similarity transformation,

$$H' = \tilde{S}^{-1} H_{FP} \tilde{S}, \quad (\text{A21})$$

with a time dependent operator \tilde{S} . Under Eq. (A21), the Fokker-Planck equation of Eq. (A7) reads

$$\partial_t \Psi = -H' \Psi, \quad (\text{A22})$$

where $\Psi = \tilde{S}^{-1} P$ with P given by Eq. (A6).

Appendix B: Supersymmetry and the mapping from classical stochastic systems to quantum mechanics

As is well known, there exists a beautiful link between stochastic classical statistical mechanics and supersymmetric quantum systems, e.g.,⁷². This connection is especially immediate for the ground state wavefunctions which are of zero energy (as indeed that of Eq. (A20)). This might lead to the impression that the results derived using the correspondence between classical dissipative systems and quantum duals are rather limited and special and was informally suggested by some to lead to un-normalizable wavefunctions if non-constant equilibrium classical states are considered. As we will explicitly elaborate on in Appendix C, the time evolution operator $\mathcal{U}(t)$ in the corresponding dual quantum problem is unitary and thus if an initial state is normalized (such as that of Eq. (A20) corresponding to an initial classical equilibrium state) then the quantum state will remain normalized at all positive times [and vice versa for a unitary evolution towards a final state of the form of Eq. (A20)]. Although the ground state of the quantum problem may be dismissed as trivial and special, the relations concerning the time evolution to states that are not of the form of Eq. (A20) are not as immediate. These relations concerning the dynamics form the core of this work. For pedagogical purposes, we very briefly review here some central notions concerning the mapping of the

Fokker Planck process to supersymmetric quantum mechanics as they, in particular, pertain to the equilibrium problem. The explicit use of supersymmetry will not be invoked in the below and the discussion will be made as simple as possible. The Hamiltonian of Eq. (5) can, for fixed $\gamma_i = \gamma$ in a simple (single-particle) one-dimensional rendition which we adopt for ease of notation, be written as

$$H = \frac{T_{cl}}{\gamma} A^\dagger A \quad (\text{B1})$$

where,

$$\begin{aligned} A^\dagger &= -\frac{\partial}{\partial x} + \frac{V'}{2T_{cl}}, \\ A &= \frac{\partial}{\partial x} + \frac{V'}{2T_{cl}}. \end{aligned} \quad (\text{B2})$$

In the higher-dimensional many body problem, the gradients are relative to each of the Cartesian coordinates of all of the particles and V is replaced by $V_N(\{\vec{x}\})$. Clearly $\gamma A^\dagger A \geq 0$ and thus if a zero-energy eigenstate of H can be found it is the ground state. Now, the square root of the classical equilibrium distribution function, i.e., the wavefunction of Eq. (A20) is clearly a null eigenstate of the operator A above. Inserting Eq. (B2) into Eq. (B1) leads to the identification of the quantum many body potential in terms of the corresponding classical potential energy $V_N(\{\vec{x}\})$. The astute reader will, up to trivial alterations, recognize these operators as the standard raising and lowering operators of the harmonic problem when V is a harmonic potential. We briefly return to this point in subsection F 2.

Appendix C: Derivation of the quantum to classical correspondence for general two operator correlators

The central role of this appendix is the derivation of Eqs.(C18,7) (or, more precisely, the derivation of Eqs. (C16, C17) that lead to Eqs. (7, C18). The sole assumption made in the below derivation of Eqs. (C16, C17, C18) is that the classical system starts from its equilibrium state and then evolves with some general (time dependent) potential $V_N(t)$. This will be mapped onto analytic continuations of the correlation and response functions of a quantum system that starts at time $t = 0$ in its ground state of Eq. (A20) and then evolved with the corresponding (time dependent) Hamiltonian $H(t)$. It is important to emphasize that we make no assumptions regarding the final (and intermediate) states. The classical (quantum) system need not stay in equilibrium (or ground state) as it evolves in time. Before detailing the derivation, we collect the basic relations discussed in

Appendix A with several new definitions,

$$P(\{x\}, t) = \langle \{x\} | P(t) \rangle, \quad (\text{C1})$$

$$P_G(\{x\}, t) = \langle \{x\} | G \rangle = \frac{e^{-V_N(\{x\})/T_{cl}}}{Z_N}, \quad (\text{C2})$$

$$H_{FP} | G \rangle = 0, \quad (\text{C3})$$

$$\langle + | \{x\} \rangle = 1, \quad (\text{C4})$$

$$H = e^{V_N/2T_{cl}} H_{FP} e^{-V_N/2T_{cl}}. \quad (\text{C5})$$

These will serve as a point of departure for the calculations in this appendix. Eq. (C1) represents a general probability distribution in bra-ket notation. Eq. (C2) defines the ground state distribution as a Boltzmann distribution in bra-ket notation. Eq. (C3) defines the ground state as the eigenvector of the Fokker-Planck Hamiltonian with zero eigenvalue. Eq. (C4) defines the state $|+\rangle$ to be the uniform state such that $|+\rangle = \int d\{x\} |+\rangle$. Lastly, Eq. (C5) can be used to find a relationship between H_{FP} and H_{FP}^\dagger .

Armed with these, we now proceed to some simple calculations. As H is Hermitian,

$$\begin{aligned} H^\dagger &= e^{-V_N/2T_{cl}} H_{FP}^\dagger e^{V_N/2T_{cl}} \\ (= H) &= e^{V_N/2T_{cl}} H_{FP} e^{-V_N/2T_{cl}}. \end{aligned} \quad (\text{C6})$$

Explicitly multiplying by $e^{V_N/2T_{cl}}$ on the left and by $e^{-V_N/2T_{cl}}$ on the right leads to

$$H_{FP}^\dagger = e^{V_N/T_{cl}} H_{FP} e^{-V_N/T_{cl}}. \quad (\text{C7})$$

We will now prove that the state $|+\rangle$ is a left eigenstate of the Fokker-Planck Hamiltonian with zero eigenvalue. Beginning with a simple extension of the definition of the ground state,

$$H_{FP} | G \rangle = 0 \rightarrow \langle G | H_{FP}^\dagger = 0. \quad (\text{C8})$$

As is evident from Eq. (C7), this is equivalent to

$$\langle G | e^{V_N/T_{cl}} H_{FP} e^{-V_N/T_{cl}} = 0, \quad (\text{C9})$$

which (from Eqs.(C2, C4)) implies that

$$Z_N^{-1} \langle + | H_{FP} e^{-V_N/T_{cl}} = 0. \quad (\text{C10})$$

This illustrates that this uniform state is a left null eigenstate for *all* Fokker-Planck Hamiltonians (i.e., $\langle + | H_{FP} = 0$).

We will now derive our central result of Eq. (7). Towards this end, we write anew the classical correlation function of Eq. (6),

$$G_{\text{classical}}(t) = \langle \mathcal{O}_1(t) \mathcal{O}_2(0) \rangle. \quad (\text{C11})$$

By Bayes' theorem, the joint probability distribution, $P(\{\vec{x}\}, \{\vec{y}\}) = P(\{\vec{x}\} | \{\vec{y}\}) P(\{\vec{y}\})$, the probability of finding coordinates $\{\vec{x}\}$ at time t and coordinates $\{\vec{y}\}$ at time 0, is given by the product of the conditional probability of finding $\{\vec{x}\}$ at time t given $\{\vec{y}\}$ at time 0 with the probability of attaining $\{\vec{y}\}$ at time $t = 0$. For a lattice system with fields ϕ at different lattice

sites (which we will briefly return to in Appendix H), the equality $P(\{\vec{x}\}, \{\vec{y}\}) = P(\{\vec{x}\}|\{\vec{y}\})P(\{\vec{y}\})$ is to be replaced by $P(\{\phi(\vec{x}, t)\}, \{\phi(\vec{x}, 0)\}) = P(\{\phi(\vec{x}, t)\}|\{\phi(\vec{x}, 0)\})P(\{\phi(\vec{x}, 0)\})$. As discussed in Appendix A, the ground state has a probability distribution given by a Boltzmann distribution $Z_N^{-1}e^{-\beta V_N(\{y\})}$ [see Eq. (C2)]. The conditional probability $P(\{x\}|\{y\})$ can be expressed in terms of the matrix element of $\mathcal{T}e^{-\int_0^t H_{FP}(t')dt'}$ (where \mathcal{T} is the time ordering operator) as this conditional P satisfies Eq. (A7). With \mathcal{O}_1 depending on the coordinates $\{y\}$ at time t and \mathcal{O}_2 on the coordinates $\{x\}$ at time $t = 0$, all this implies the form of the expectation value of Eq. (C11),

$$\begin{aligned} G_{\text{classical}}(t) &= \int d^{dN}x d^{dN}y \mathcal{O}_1 P(\{\vec{x}\}, \{\vec{y}\}) \mathcal{O}_2 \\ &= \int d^{dN}x d^{dN}y \mathcal{O}_1 P(\{\vec{x}\}|\{\vec{y}\}) \mathcal{O}_2 P(\{\vec{y}\}) \\ &= \int d^{dN}x d^{dN}y \mathcal{O}_1 \langle \{\vec{x}\} | \mathcal{T}e^{-\int_0^t H_{FP}(t')dt'} | \{\vec{y}\} \rangle \mathcal{O}_2 \\ &\quad \times \frac{e^{-\beta V_N(\{\vec{y}\})}}{Z_N}. \end{aligned}$$

It is important to re-iterate and emphasize yet again that, in the last line above, we only assume that *the initial state* ($\{\vec{y}\}$ at time $t = 0$) *is in thermal equilibrium*. *The system need not be in thermal equilibrium at positive times*. As stated earlier, this is the sole assumption made in this derivation for general time dependent systems with dynamical V_N (and thus a time dependent Fokker-Planck operators). A similar derivation would hold *mutatis mutandis* when the system is initially out of equilibrium and is in equilibrium in its final state.

Eq. (C4) asserts that $\int d^{dN}x \langle \{\vec{x}\} | = \langle + |$. Invoking this along with Eq. (C2), we have

$$G_{\text{classical}}(t) = \langle + | \mathcal{O}_1 \mathcal{T}e^{-\int_0^t H_{FP}(t')dt'} \mathcal{O}_2 | G \rangle. \quad (\text{C12})$$

As is evident from Eq. (C10), inserting an exponentiation of H_{FP} to the right of the state $\langle + |$ leads to a multiplication by unity. Thus Eq. (C12) can be rewritten as

$$\begin{aligned} G_{\text{classical}}(t) &= \langle + | \mathcal{T}e^{\int_0^t H_{FP}(t')dt'} \mathcal{O}_1 \\ &\quad \times \mathcal{T}e^{-\int_0^t H_{FP}(t')dt'} \mathcal{O}_2 | G \rangle. \end{aligned} \quad (\text{C13})$$

With the aid of Eq. (C5), we can express this quantity in terms of the quantum Hamiltonian H instead of H_{FP} ,

$$\begin{aligned} G_{\text{classical}}(t) &= \langle + | e^{-V_N/(2T_{cl})} \mathcal{T}e^{\int_0^t H(t')dt'} \mathcal{O}_1 \\ &\quad \times \mathcal{T}e^{-\int_0^t H(t')dt'} e^{V_N/(2T_{cl})} \mathcal{O}_2 | G \rangle. \end{aligned} \quad (\text{C14})$$

Rather explicitly multiplying and dividing by $\sqrt{Z_N}$,

$$\begin{aligned} G_{\text{classical}}(t) &= \langle + | \frac{e^{-V_N/(2T_{cl})}}{\sqrt{Z_N}} \mathcal{T}e^{\int_0^t H(t')dt'} \mathcal{O}_1 \mathcal{T}e^{-\int_0^t H(t')dt'} \\ &\quad \times \mathcal{O}_2 \sqrt{Z_N} e^{V_N/(2T_{cl})} | G \rangle \end{aligned} \quad (\text{C15})$$

As discussed in Appendix A (in particular, Eq. (A20)), the ground state of the quantum system is given by $|0\rangle =$

$\sqrt{Z_N} e^{V_N/(2T_{cl})} |G\rangle$. Further invoking Eqs. (C2, C4), we can rewrite Eq. (C15) as

$$\begin{aligned} G_{\text{classical}}(t) &= \langle 0 | \mathcal{T}e^{\int_0^t H(t')dt'} \mathcal{O}_1 \\ &\quad \times \mathcal{T}e^{-\int_0^t H(t')dt'} \mathcal{O}_2 | 0 \rangle. \end{aligned} \quad (\text{C16})$$

Note that in this equation, $|0\rangle$ is the ground state of the system defined by the quantum Hamiltonian H . Our results above are general. We will shortly use Eq. (C16) in order to relate it to correlations in the quantum system. Under the exchange of t by (it) , the reader may recognize Eq. (C16) as a correlation function in the quantum system. One very simple point which is worth emphasizing is that not only the ground state of Eq. (A20) is trivially normalized but, of course, any state formed by the evolution with the unitary time dependent time ordered exponential $\mathcal{U}(t) = e^{-i\int_0^t H(t')dt'}$.

In the quantum arena, it is clear that for a system initially prepared in the ground state $|0\rangle$ and then evolved with some Hamiltonian $H(t)$, the corresponding correlation function is given by

$$\begin{aligned} G_{\text{Quantum}}(t) &= \langle 0 | \mathcal{T}e^{i\int_0^t H(t')dt'} \mathcal{O}_1 \mathcal{T}e^{-i\int_0^t H(t')dt'} \\ &\quad \times \mathcal{O}_2 | 0 \rangle. \end{aligned} \quad (\text{C17})$$

By comparing Eq. (C16) with Eq. (C17), it is seen that

$$G_{\text{Quantum}}(t) = G_{\text{classical}}(it). \quad (\text{C18})$$

This fundamental identity establishes the connection between the overdamped Langevin equation of a classical particle at finite temperature and the Schrödinger equation of the dual quantum Hamiltonian. We suspect that related to this result is the fluctuation-dissipation theorem that relates correlation functions with the expectation value of time-ordered products in equilibrium, see for example Ref. 15 (Chap. 13).

We next turn to the quantum response function R_{Quantum} that monitors the change in the average value of \mathcal{O}_1 as a result of a perturbation \mathcal{O}_2 . Specifically, we consider the Hamiltonian

$$H_{\text{tot}} = H + H', \quad (\text{C19})$$

where H' is a small perturbation which can be expressed as $H' = -\lambda \mathcal{O}_2$. We next review the standard protocol for computing the lowest order deviation

$$\delta \langle \mathcal{O}_1(t) \rangle = \langle \mathcal{O}_1(t) \rangle_\lambda - \langle \mathcal{O}_1(t) \rangle_0, \quad (\text{C20})$$

which we will evaluate within the ground state $|0\rangle$. This deviation is readily computed within the interaction picture where we evolve with the time ordered exponential $\mathcal{T} \exp(-iH't)$,

$$\begin{aligned} \langle \mathcal{O}_1(t) \rangle_\lambda &= \left\langle \left(1 - i \int_0^t dt' \lambda(t') \mathcal{O}_2(t') \right) \mathcal{O}_1(t) \right. \\ &\quad \times \left. \left(1 + i \int_0^t dt' \lambda(t') \mathcal{O}_2(t') \right) \right\rangle. \end{aligned} \quad (\text{C21})$$

Collecting terms to lowest order,

$$\begin{aligned}\delta\langle\mathcal{O}_1(t)\rangle &\approx i \int_0^t dt' \lambda(t') \langle[\mathcal{O}_1(t), \mathcal{O}_2(t')]\rangle \\ &= i \int_0^\infty d\tau \lambda(t-\tau) \langle[\mathcal{O}_1(\tau), \mathcal{O}_2(0)]\rangle \\ &\equiv \int_{-\infty}^\infty d\tau \lambda(t-\tau) R_{\text{Quantum}}(\tau). \quad (\text{C22})\end{aligned}$$

As $\mathcal{O}_1(t) = \mathcal{T}e^{-i\int_0^t H(t')dt'} \mathcal{O}_1 \mathcal{T}e^{i\int_0^t H(t')dt'}$, from the last line of Eq. (C22), the quantum response function

$$\begin{aligned}R_{\text{Quantum}}(t) &= i\Theta(t)\langle 0 | [\mathcal{T}e^{i\int_0^t H(t')dt'} \mathcal{O}_1 \\ &\quad \times \mathcal{T}e^{-i\int_0^t H(t')dt'}, \mathcal{O}_2] | 0 \rangle. \quad (\text{C23})\end{aligned}$$

Comparing Eqs. (C16, C23), we derive Eq.(7) by further expanding it to get the imaginary part of the analytically continued classical correlation function,

$$\begin{aligned}R_{\text{Quantum}}(t) &= i\Theta(t)(G_{\text{classical}}(it) - G_{\text{classical}}^*(it)) \\ &= -2\Theta(t)\Im G_{\text{classical}}(it). \quad (\text{C24})\end{aligned}$$

Appendix D: Arbitrary Number of Time-Ordered Operators

A derivation similar to that in Appendix C can be performed for a correlation function involving an arbitrary number of operators. In the classical case, the correlation function takes the form of

$$G_{cl} = \langle \mathcal{O}_1(t_1) \mathcal{O}_2(t_2) \dots \mathcal{O}_n(t_n) \rangle, \quad (\text{D1})$$

where \mathcal{O}_i are arbitrary operators and $t_1 < t_2 < \dots < t_n$.

Similar to our calculations in Appendix C, by Bayes' theorem this correlation function is given by

$$\begin{aligned}&\int d^{dN}x_1 d^{dN}x_2 \dots d^{dN}x_n \mathcal{O}_n \\ &\times \langle \{\vec{x}_n\} | \mathcal{T}e^{-\int_{t_{n-1}}^{t_n} H_{FP}(t')dt'} | \{\vec{x}_{n-1}\} \rangle \mathcal{O}_{n-1} \\ &\dots \langle \{\vec{x}_2\} | \mathcal{T}e^{-\int_{t_1}^{t_2} H_{FP}(t')dt'} | \{\vec{x}_1\} \rangle \mathcal{O}_1 \frac{e^{-\beta U_N(\{\vec{x}_1\})}}{Z_N}.\end{aligned}$$

Invoking identity matrix insertions and integrations over complete eigensets of states as before, this reduces to

$$\begin{aligned}\langle + | \mathcal{T}e^{-\int_{t_n}^{t_1} H_{FP}(t')dt'} \mathcal{O}_n \mathcal{T}e^{-\int_{t_{n-1}}^{t_n} H_{FP}(t')dt'} \mathcal{O}_{n-1} \\ \dots \mathcal{T}e^{-\int_{t_1}^{t_2} H_{FP}(t')dt'} \mathcal{O}_1 | G \rangle. \quad (\text{D2})\end{aligned}$$

Transforming to the quantum Hamiltonian $H(t)$ and its respective ground state at time $t = 0$ yields

$$\begin{aligned}\langle 0 | \mathcal{T}e^{-\int_{t_n}^{t_1} H(t')dt'} \mathcal{O}_n \mathcal{T}e^{-\int_{t_{n-1}}^{t_n} H(t')dt'} \mathcal{O}_{n-1} \\ \dots \mathcal{T}e^{-\int_{t_1}^{t_2} H(t')dt'} \mathcal{O}_1 | 0 \rangle. \quad (\text{D3})\end{aligned}$$

The remainder of the derivation is similar to that in Appendix C. In order to go from the classical to the quantum system, we replace the times $\{t_a\}_{a=1}^n$ by $\{it_a\}_{a=1}^n$ to go from the classical to the many body quantum problem.

Appendix E: Analytic continuation of classical stretched exponentials

In Appendices C and D, we explicitly illustrated how a Wick type rotation $t \rightarrow it$ relates time dependent correlation functions in the viscous classical systems to those in the dual quantum many body theories. For the sake of clarity, we explicitly discuss how the analytic continuation of the classical correlation function should be performed in some simple yet empirically relevant cases for dynamical response functions in viscous classical systems wherein Eq. (8) describes the dynamical response. In Eq. (9), we provided the quantum dual to exponentially stretched classical dynamics. In this brief appendix, we describe how this result is derived and outline how analytic continuations for other response functions of classical viscous systems may be analytically continued following such a Wick type rotation. We thank Carl Bender for a quick tutorial on aspects of Stokes wedges on which this brief appendix heavily relies.

If $G_{\text{classical}}(t) = \sum_n B_n e^{-t/\tau_n}$ then, trivially, the quantum response function will be uniquely defined and given by $R_Q = \sum_n B_n \cos(t/\tau_n)$. The same applies, of course, for distributions of modes (whence the discrete sum over overdamped classical modes n is replaced by an integral with some density of modes $f(\tau)$). In the limit of an infinite number of modes,

$$G_{\text{classical}}(t) = \int_0^\infty d\tau' f(\tau') \exp(-t/\tau'), \quad (\text{E1})$$

with P a distribution that generally is no longer a sum of Dirac delta functions. As we will further elaborate on in the appendices [see, in particular, Appendix C], $G_{\text{quantum}}(t) = G_{\text{classical}}(it)$. Thus,

$$G_{\text{Quantum}}(t) = \int_0^\infty d\tau' f(\tau') \exp(-it/\tau'). \quad (\text{E2})$$

In the complex τ' plane, for any “well-behaved” function $f(\tau')$ that is localized in a region of positive finite τ' , the integral of Eq. (E1) may be performed along any contour connecting the origin and the $\tau' = \infty$ along the real line such that the contour lies exclusively in the right half complex plane (the pertinent Stokes wedge in this case) of a positive real component of τ' , i.e., $\Re\{\tau'\} \geq 0$. Of particular interest to us is the stretched exponential form given by $G_{\text{classical}}(t) = A \exp[-(t/\tau)^a]$. Now, in performing the substitution $t \rightarrow it$ to implement the transformation from Eq. (E1) to Eq. (E2), we perform the rotation

$$t = te^{i\varphi}, \quad \varphi : 0 \longrightarrow \varphi_{\text{final}}, \quad (\text{E3})$$

with $\varphi_{final} = \pi(4n+1)/2$ where n is an integer. The integral of Eq. (E2) remains well-defined in the top complex half-plane of a positive real part of τ' , i.e., $\Re\{\tau'\} \geq 0$ (the rotated counterpart of the original Stokes wedge). If φ is varied continuously from 0 to $\pi/2$, there remain contours from $\tau' = 0$ to $\tau' = \infty$ that appear in the original Stokes wedge of $\Re\{\tau'\} \geq 0$ that pass exclusively through the region $\Im\{\tau'\} \geq 0$; the integrals along these contours can be analytically continued when φ is continuously increased from 0 to $\pi/2$. Thus, we may perform the rotation of Eq. (E3) continuously increasing φ to represent i as $e^{i\pi/2}$ and replace $t \rightarrow te^{i\pi/2}$ in the argument of $G_{classical}(t)$. This is what we have done in Eq. (9). For other choices of n for $\varphi_{final} = \pi(4n+1)/2$ as we continuously vary φ from its initial value of zero, there will always appear situations where the original Stokes wedge will have no overlap with its rotated counterpart. Thus, the substitution of $t \rightarrow te^{i\pi/2}$ in the argument of $G_{classical}(t)$ is the only one that may be implemented out of the possible choices in Eq. (E3) in order to evaluate the integral of Eq. (E2). This forms the correct analytic continuation of the original real time correlation function of $G_{classical}(t)$ of Eq. (E1).

Appendix F: Simple examples of classical to quantum correspondence and some aspects

To elucidate some aspects of the known mapping between classical dissipative and quantum systems, we discuss several extremely simple examples in d spatial dimensions.

1. Non-interacting particles

In the case of a system with a zero potential everywhere, the quantum ground state wave-function is a constant in real space. That this is so is seen by our mapping as, by Eq.(A20), the classical potential energy vanishes everywhere. By Eq. (5), the same also occurs for the quantum potential which is everywhere zero.

a. Zero energy bound state

For a short range attractive potential the zero energy eigenstate outside the potential, up to volume normalization factors, given by

$$\Psi_0(\vec{x}) = \frac{A}{|\vec{x}|^{d-2}}. \quad (F1)$$

Invoking Eq. (A20), we see that, in this case,

$$V_N^{\text{free}}(\{\vec{x}\}) = 2T_{cl}(d-2) \ln |\vec{x}|. \quad (F2)$$

Indeed substituting Eq. (F2) into Eq. (5) and recalling that, in its scalar “S-wave” (or “ $\ell = 0$ ”) representation,

the Laplacian is given by $\nabla^2 = \frac{d^2}{dr^2} + \frac{d-1}{r} \frac{d}{dr}$, it is readily verified, as it must self-consistently be, that the corresponding quantum potential $V_{\text{Quantum}} = 0$ in the region outside the range of the interaction.

2. Harmonic oscillator systems

As seen by Eq. (5), classical systems with harmonic potentials V_N map onto quantum systems with similar (up to innocuous shifts) harmonic potentials V_{Quantum} . That this must be so is readily seen as the ground state Ψ_0 of simple quantum harmonic potentials is given by a Gaussian. Using Eq. (A20), we see that this indeed relates to a harmonic classical potential V_N as it must. As can be further seen from Eq. (B2), in the case of harmonic classical systems, the operators A and A^\dagger are trivially related to the raising and lowering operators in the quantum harmonic problem (and indeed the Gaussian form of the ground state can, as is very well known, be seen from the requirement that the annihilation operator must yield zero when acting on the ground state).

3. Localization and delocalization

A general localization in the quantum problem will appear whenever a corresponding spatial localization occurs in the classical system. That is, if the probability of finding a classical particle outside a distance R from its initial location, e.g., the origin,

$$h(R) = \int_{x>R} d^d x P^{\text{equil}}(\vec{x}) \quad (F3)$$

tends to zero, i.e., $h(R) \sim \exp(-R/R_0)$ for $R \gg R_0$, then a localized or bound state is also present in the quantum dual. Here $x = |\vec{x}|$. In other words, the particle is localized within a range R . From Eqs. (A13,A20), in a system with pair interactions, a localized ground state may appear only if

$$\lim_{r \rightarrow \infty} \frac{dv^{\text{pair}}(r)}{dr} > 0. \quad (F4)$$

4. Scaling invariance of time and space

As is well known, for a homogeneous classical potential $V_N(\{\vec{x}\})$ which scales as a power (say, p) of the spatial coordinates $|\vec{x}|$, the equations of motion are invariant under a simultaneous rescaling of the time coordinates. This analysis is typically done for inertial systems. When replicated for the over damped system of Eq. (3), we find that

$$\vec{x}_i \rightarrow a\vec{x}_i, \quad t \rightarrow bt, \quad (F5)$$

where b plays the role of λ and a plays the role of $\lambda^{1/z}$ from before, leads to an invariance of Eq. (3) if $b = a^{2-p}$.

By contrast, in the corresponding quantum problem of the Schrödinger equation with the Hamiltonian of Eq. (5), a scaling such as that of Eq.(F5) is possible only for a single case: that of a potential $V_N(\{\vec{x}\})$ that is a logarithmic function of its arguments (or a constant). For this particular case, we find that $b = a^2$. Correspondingly, akin to subsection F 1, for this particular case, the time scales as $t \sim |x|^2$ as in diffusion or the free particle quantum problem.

5. Relation between the classical and quantum potentials in the eikonal approximation to the Schrödinger equation

Below we briefly review the eikonal approximation and then discuss its relation to the connection between the classical and quantum many body potentials as seen in Eqs. (5, A20). Towards this end, we write the wavefunction as a function of only the phase, the eikonal approximation [as throughout, we set $\hbar = 1$],

$$\Psi_0 = A e^{iS}, \quad (\text{F6})$$

and substitute this into the Schrödinger equation with the Hamiltonian in the second line of Eq. (5) then we will arrive at

$$\begin{aligned} \frac{1}{2m} \sum_i (\vec{\nabla}_i S)^2 + \mathcal{V}_{\text{Quantum}}(\{\vec{x}\}) + \frac{\partial S}{\partial t} \\ = \frac{i}{2m} \sum_i \nabla_i^2 S. \end{aligned} \quad (\text{F7})$$

For time independent solutions, $\frac{\partial S}{\partial t} = 0$ and Eq. (F7) rather trivially becomes

$$\mathcal{V}_{\text{Quantum}}(\{\vec{x}\}) = \sum_i \left[\frac{i}{2m} \nabla_i^2 S - \frac{1}{2m} (\vec{\nabla}_i S)^2 \right]. \quad (\text{F8})$$

If we now invoke the correspondence $iS \leftrightarrow -\beta V_N/2$ then Eq. (F6) will transform into Eq. (A20) and, similarly, Eq. (F8) will become Eq. (5) relating the quantum potential energy $\mathcal{V}_{\text{Quantum}}$ to the classical potential energy V_N .

Appendix G: Slater-Jastrow forms

The general results presented thus far may, in some instances, be generalized to describe fermions. A limited extent extension is the one concerning the evolution starting off from an initial Slater-Jastrow type fermionic wavefunction. As we have emphasized earlier, if the V_{ij} in Eq. (A10) are symmetric under the exchange of i and j , the resulting wavefunction obeys Bose statistics. This symmetry is maintained for the ground state as it is a Jastrow function given by Eq. (A20). Fermionic wavefunctions are afforded by the product of the symmetric boson ground state and an antisymmetric term,

$$\Psi_F = \Psi_0 \chi. \quad (\text{G1})$$

The function χ can take any antisymmetric form. For simplicity, we choose it to be a Slater determinant of the form

$$\chi = \frac{1}{\Omega^{N/2}} \frac{1}{\sqrt{N!}} \begin{vmatrix} e^{i\vec{k}_1 \cdot \vec{r}_1} & e^{i\vec{k}_1 \cdot \vec{r}_2} & \dots & e^{i\vec{k}_1 \cdot \vec{r}_N} \\ e^{i\vec{k}_2 \cdot \vec{r}_1} & e^{i\vec{k}_2 \cdot \vec{r}_2} & \dots & e^{i\vec{k}_2 \cdot \vec{r}_N} \\ \vdots & \vdots & & \vdots \\ e^{i\vec{k}_N \cdot \vec{r}_1} & e^{i\vec{k}_N \cdot \vec{r}_2} & \dots & e^{i\vec{k}_N \cdot \vec{r}_N} \end{vmatrix}, \quad (\text{G2})$$

with Ω the volume of the system.

$$\text{As } i\partial_t \Psi_0 = H \Psi_0,$$

$$i\partial_t(\Psi_0 \chi) = \chi(i\partial_t \Psi_0) + \Psi_0(i\partial_t \chi) \quad (\text{G3})$$

$$= \chi H \Psi_0 + \Psi_0 E_{\text{Slater}} \chi, \quad (\text{G4})$$

where E_{Slater} is the energy of the free particle system described by χ .

$$H(\Psi_0 \chi) = (T_0 + \mathcal{V}_{\text{Quantum}})(\Psi_0 \chi). \quad (\text{G5})$$

The potential energy operator leads to $\chi(\mathcal{V}_{\text{Quantum}} \Psi_0)$. The kinetic energy operator T generates three terms of, respectively, the forms $\sum_a (\nabla_a^2 \chi) \psi^0$, $\sum_a (\nabla_a \chi) \cdot (\nabla_a \psi^0)$, and $\sum_a \chi (\nabla_a^2 \psi)$. The first and the last terms represent the term proportional to E_{Slater} and the original bosonic kinetic energy respectively. The second term, that of the mixed gradients, is proportional to $\sum_a \vec{k}_a$. For a system invariant under parity, this sum vanishes. Up to an innocuous phase factor, the evolution given an initial fermionic wavefunction of Eq. (G1), will be thus identical to that with the bosonic wavefunction Ψ_0 and all correlation functions will be identical to those which we earlier computed for the bosonic system. That is, the general time dependent correlation functions given an initial fermionic state of Eqs.(A20,G1) will adhere to the general $t \rightarrow it$ rule which we detailed in earlier sections. A notable difference by comparison to the bosonic case, however, is that the wavefunction of Eq. (G1) at an initial (or at a final) time is, generally, not a ground state of the Hamiltonian H .

Appendix H: Extensions: complex wavefunctions and lattice systems

We now briefly suggest and elaborate on several extensions of our calculations thus far. We will illustrate and suggest how our results may hold for (i) several lattice extensions as well as (ii) general systems with complex wavefunctions. This latter case allows us to go from a given quantum mechanical problem to a corresponding classical one.

(i) *Lattice systems.* If we replace Eq. (3) by

$$\gamma_i \frac{d\phi_i}{dt} = -\frac{\delta}{\delta \phi_i} V_N(\phi_1, \dots, \phi_N) + \eta_i(t), \quad (\text{H1})$$

to describe a classical lattice system with fields ϕ_i at the various lattice points i , then trivially replicating all of our calculations thus far with the exchange $\tilde{x}_i \rightarrow \phi_i$ (including in all gradient or variational derivative operators), we will arrive at a corresponding quantum lattice system.

(ii) *Complex wavefunctions.* The similarity transformation of Eqs. (A9, A19, A21) captures a simple mathematical identity between the generalized probability distribution of a classical system, obeying the Fokker-Planck dynamics with an operator H_{FP} (Eq. (A7)), and the wavefunction obeying the Schrödinger equation of the quantum dual Hamiltonian H . Given this relation, it is possible to, formally, consider extensions in which the function evolving with the Fokker-Planck dynamics need not be a probability distribution. Most of our results concerning temporal correlations may hold under such a generalized interchange if we invoke Eq. (A20) to define (when given a quantum problem) a corresponding classical system which need not be a bona fide physical Boltzmann distribution as in the scalar bosonic sys-

tems which we primarily focused on thus far and employ $|\Psi_0|^2$ as the initial (or final) time weight in the multiple time classical correlation function of Appendix C. In appendix F 5, we examined a formally imaginary counterpart to V_N and explicitly demonstrated how it leads to standard results. Thus, given a quantum wavefunction, we may consider its logarithm to correspond to a classical potential V_N . Wavefunctions of spinless Fermi systems cannot be purely positive and for these complex (as well as divergent) potentials will formally arise. There may be subtleties however in our imaginary time ($t \rightarrow it$) analytic continuations when V_N is not purely real (and the system effectively not purely dissipative) which are more complex than those which we invoked thus far in our analysis of real V_N which led to response functions of pure damped modes and their superpositions such as those which we encountered in Eq. (8) [see also Appendix E]. Physically, these are related to analogs of classical systems with instantons and tunneling events (the behavior for the pure dissipative system) when these further exhibit non-damped oscillatory behavior.

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